Counting Intersections

Note Title 15/01/2006

Given natural numbers M, N, how many unit squares does the (real) line

$$x, y :: \quad Nx = My$$

intersect, starting from $(0,0)$ and ending at (M,N)?

(Note: x and y range over real numbers. All other variables in this document are natural numbers.)

The solution obviously involves counting the number of squares — i.e. an algorithm.

Suppose p, q are numbers and consider the square

$$x,y :: \quad p-1 < x \leq p \wedge q-1 < y \leq q \, .$$

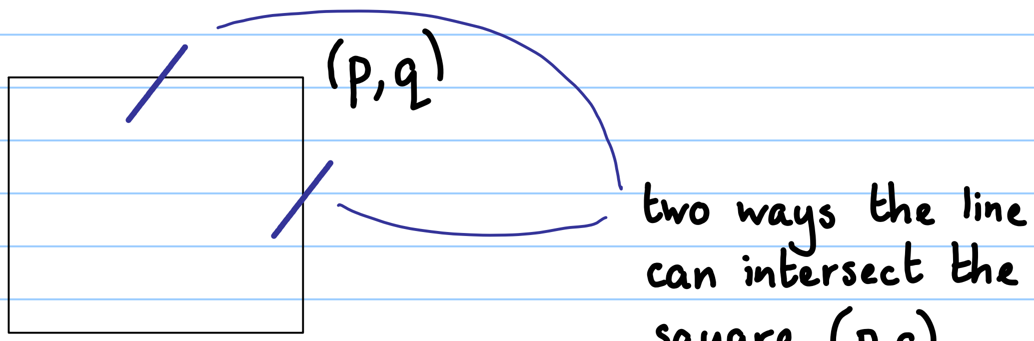
For brevity, we call this *the square* (p,q).

The line intersects the square (p,q) equivales the line intersects the line

$$x,y :: \quad p-1 < x \leq p \wedge y = q$$

or the line intersects the line

$$x,y :: \quad x = p \wedge q-1 < y \leq q \, .$$



Now,

$$p-1 < x \leq p \wedge y = q \wedge Nx = My$$

$= \quad \{ \text{ Leibniz, monotonicity of product } \}$

$$Np - N < Mq \leq Np \wedge y = q \wedge Nx = My$$

Consequently,

$$\langle \exists x,y :: p-1 < x \leq p \wedge y = q \wedge Nx = My \rangle$$

$= \quad \{ \; x,y := Mq/N, q \; \}$

$$Np - N < Mq \leq Np \quad .$$

Similarly,

$$\langle \exists x,y :: x = p \wedge q-1 < y \leq q \wedge Nx = My \rangle$$

$=$

$$Mq - M < Np \leq Mq \quad .$$

We conclude that

"the square (p,q) intersects the line" is

$$(Np - N < Mq \leq Np) \vee (Mq - M < Np \leq Mq) \, .$$

Using this, an algorithm to count the number of squares intersected by the line is :

$p, q := 0, 0$; $c := 0$

{Invariant:

the square (p,q) intersects the line

$\wedge$ c counts the intersecting squares

below and to the left of (p,q) }

do

$p < M \vee q < N \rightarrow c := c+1$

; if

$Np < Mq \rightarrow p := p+1$

$\square$ $Np = Mq \rightarrow p, q := p+1, q+1$

$\square$ $Np > Mq \rightarrow q := q+1$

fi

od

Our next task is to establish an invariant relation between c and (p,q).

We note that c is incremented by 1 when exactly one of p or q is incremented, or both p and q are incremented. This suggests the invariant

$$c = p+q-d$$

where

$$d = \langle \sum p',q' : \; p'<p \wedge q'<q \wedge Np'=Mq' : 1 \rangle$$

(In words, d counts the number of times the given line intersects points (p',q').)

If we add the computation of d to the algorithm, we get:

$p, q := 0, 0 \;;\; c, d := 0, 0$

$\{$Invariant: $c = p + q - d$

$\wedge$ the square (p,q) intersects the line

$\wedge$ c counts the intersecting squares below and to the left of (p,q)

$\wedge$ $d = \langle \Sigma\, p', q' : p' < p \wedge q' < q \wedge Np' = Mq' : 1 \rangle$ $\}$

do

$p < M \vee q < N \rightarrow c := c + 1$

$\quad ;$ if

$\qquad Np - N < Mq < Np \rightarrow q := q + 1$

$\qquad \Box\; Np = Mq \rightarrow p, q, d := p+1, q+1, d+1$

$\qquad \Box\; Mq - M < Np < Mq \rightarrow p := p + 1$

$\quad$ fi

od

On termination of the algorithm,

$$p = M \wedge q = N \wedge c = p+q-d .$$

We conclude that the required count is

$$M+N-d$$

where

$$d = \langle \Sigma p,q : p<M \wedge q<N \wedge N_q = M_p : 1 \rangle .$$

The final task is to evaluate d.

Now,

$$Np = Mq$$

$=$ { let $m \upharpoonleft n$ denote $\dfrac{m}{m \gcd n}$ }

$$(N \upharpoonleft M) \times p = (M \upharpoonleft N) \times q$$

$=$ { by definition of gcd,
$N \upharpoonleft M$ and $M \upharpoonleft N$ are coprime }

$$\langle \exists k :: p = k \times (M \upharpoonleft N) \wedge q = k \times (N \upharpoonleft M) \rangle .$$

So,

$$
\begin{array}{ll}
 & d \\
= & \quad \{ \text{ definition } \} \\
 & \langle \Sigma' p,q : p < M \wedge q < N \wedge Nq = Mp : 1 \rangle \\
= & \quad \{ \text{ above calculation } \} \\
 & \langle \Sigma' p,q : p < M \wedge q < N \\
 & \qquad \wedge \langle \exists k :: p = k \times (M \upharpoonright N) \wedge q = k \times (N \upharpoonright M) \rangle \\
 & \qquad : 1 \rangle \\
= & \quad \{ \text{ range splitting, one-pt. rule } \} \\
 & \langle \Sigma' k : k \times (M \upharpoonright N) < M \wedge k \times (N \upharpoonright M) < N : 1 \rangle \\
= & \quad \{ \text{ by definition of } \upharpoonright, \\
 & \qquad \dfrac{M}{M \upharpoonright N} = M \gcd N = \dfrac{N}{N \upharpoonright M} \quad \} \\
 & \langle \Sigma' k : k < M \gcd N : 1 \rangle \\
= & \quad \{ \text{ trivial property of summation } \} \\
 & M \gcd N .
\end{array}
$$

Substituting for d, we conclude that the number of squares intersected by the line is

$$\boxed{M+N-(M \gcd N)} \ .$$

Concluding Remark

The problem is a good illustration of *algorithmic problem solving*. It is obviously about an algorithm —how the squares are counted— and equational reasoning with quantifiers. I am grateful to Joe Kyle for bringing the problem to my attention.

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